Feynman Exercises Lecture Challenge

Dale Lukas Peterson <hazelnusse@gmail.com>

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Restrictions on Solution Method

In one of the Review lectures Feynman gave to his freshman students, just before their first big exam, he advised them as follows (copied from Feynman's Tips on Physics, a problem-solving supplement to The Feynman Lectures on Physics):

> Now, all these things you can feel. You don't have to feel them; you can work them out by making diagrams and calculations, but as problems get more and more difficult, and as you try to understand nature in more and more complicated situations, the more you can guess at, feel, and understand without actually calculating, the much better off you are! So that's what you should practice doing on the various problems: when you have time somewhere, and you're not worried about getting the answer for a quiz or something, look the problem over and see if you can understand the way it behaves, roughly, when you change some of the numbers.

The challenge is to solve the problem given below (originally homework for FLP Vol. I, chapter 23) in the spirit of Feynman's advice, above. It must be solved without using any calculus or differential equations or integral equations or difference equations, etc., without iterative numerical methods, nor any such fancy mathematical tricks! You may use only algebra, geometry, trigonometry, dimensional analysis, and Newtonian mechanics, in your solution, which should be guided by your physical intuition (however note: all intuitions used in solutions must be justified)! Your answer does not have to be exact, but it should at least be a very close approximation. And be sure to show all your work! Here is the problem:

Problem Statement

The pivot point of a simple pendulum having a natural period of 1.00 second is moved laterally in a sinusoidal motion with an amplitude 1.00 cm and period 1.10 seconds. With what amplitude should the pendulum bob swing after a steady motion is attained?

Solution

The first step in the solution process is to obtain the equation of motion. To this end, I introduce a few symbols to represent physical quantities: m, l, g, being the mass of the pendulum, the length from the pendulum pivot to the mass center, and the gravitational constant. Without loss of generality, I assume a simple point mass pendulum with a massless rod, and no dissipation of any kind.

The velocity and acceleration of the pendulum mass relative to an inertial frame are

$$\begin{aligned}
\mathbf{v} &= \dot{x}\hat{\mathbf{i}} + \dot{\theta}\hat{\mathbf{j}} \times \left(l\sin\theta\hat{\mathbf{i}} + l\cos\theta\hat{\mathbf{k}}\right) \\
&= \left(\dot{x} + l\dot{\theta}\cos\theta\right)\hat{\mathbf{i}} + \left(-l\dot{\theta}\sin\theta\right)\hat{\mathbf{k}} \\
\mathbf{a} &= (\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta)\hat{\mathbf{i}} + (-l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta)\hat{\mathbf{k}}
\end{aligned}$$

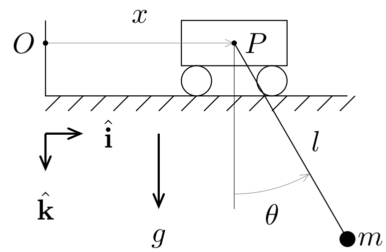

Figure 1: Pendulum with horizontally translating pivot. $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$ points out of the page.

Two forces act on the mass; the gravitational force $\mathbf{F}_g$ and the force of the rod $\mathbf{F}_r$:

$$\mathbf{F}_g = mg\hat{\mathbf{k}} \qquad \mathbf{F}_r = -F\sin\theta\hat{\mathbf{i}} - F\cos\theta\hat{\mathbf{k}}$$

where F is a function of time. Newton's 2nd law yields:

$$mg\hat{\mathbf{k}} - F\sin\theta\hat{\mathbf{i}} - F\cos\theta\hat{\mathbf{k}} = m(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta)\hat{\mathbf{i}} + m(-l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta)\hat{\mathbf{k}}$$

Dotting this equation with $\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{k}}$, dividing through by ml and rearranging, yields:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{0.01}{l}\left(\frac{2\pi}{1.1}\right)^2 \sin\left(\frac{2\pi}{1.1}t\right)\cos\theta$$

Notice how dotting the Newton's equation into the direction perpendicular to the rod is a convenient way to eliminate the constraint force $\mathbf{F}_r$. Linearizing about the downwards position $\theta = 0$, we have $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, which yields the following second order linear differential equation:

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{0.01}{l}\left(\frac{2\pi}{1.1}\right)^2 \sin\left(\frac{2\pi}{1.1}t\right)$$

Defining $\omega_n^2 = \frac{g}{l}$ and $f_0 = \frac{0.01}{l}\left(\frac{2\pi}{1.1}\right)^2$, and $\omega = \frac{2\pi}{1.1}$, we can rewrite the equation of motion as

$$\ddot{\theta} + \omega_n^2\theta = f_0\sin\omega t$$

Assuming a particular solution of the form $\theta_p(t) = \Theta\sin\omega t$, differentiating twice with respect to time, and substituting into the equation of motion yields:

$$(-\omega^2 + \omega_n^2)\Theta\sin(\omega t) = f_0\sin\omega t$$

which implies that the amplitude of swing of the pendulum bob after steady motion is attained is:

$$\begin{aligned}\Theta &= \frac{f_0}{\omega_n^2 - \omega^2} \\ &= \frac{\frac{0.01}{l}\left(\frac{2\pi}{1.1}\right)^2}{(2\pi)^2 - (\frac{2\pi}{1.1})^2} \\ &= \frac{0.047619}{l}\end{aligned}$$

Where we made use of the fact that the natural frequency is related to the natural period of oscillation by $\omega_n = 2\pi/T = 2\pi$ rad/s.

Note that this solution depends on the length of the rod. However, if we assume Earth's gravitational constant $g = 9.81$ m/s/s, then from $g/l = \omega_n^2$ we can calculate that $l = 0.2485$ m. Under this assumption, the amplitude of oscillations is then:

$$\begin{aligned}\Theta &= 0.1916 \text{ rad} \\ &= 10.98 \text{ deg}\end{aligned}$$

This allows us to find the horizontal and vertical oscillations of

$$
\begin{aligned}
X &= l \sin \Theta + 11\text{mm} \\
&= 57.3\text{mm} \\
Y &= \frac{l}{2}(1 - \cos \Theta) \\
&= 2.27\text{mm}
\end{aligned}
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